Klein Tunnelling and The Klein Paradox

A Calogeracos

(acal@hol.gr)

NCA Research Consultants, PO Box 61147,Maroussi 151 22, Greece

N Dombey

(normand@sussex.ac.uk)

Centre for Theoretical Physics, University of Sussex, UK

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The Klein paradox is reassessed by considering the properties of a finite square well or barrier in the Dirac equation. It is shown that spontaneous positron emission occurs for a well if the potential is strong enough. The vacuum charge and lifetime of the well are estimated. If the well is wide enough, a seemingly constant current is emitted. These phenomena are transient whereas the tunnelling first calculated by Klein is time-independent. Klein tunnelling is a property of relativistic wave equations, not necessarily connected to particle emission. The Coulomb potential is investigated in this context: it is shown that a heavy nucleus of sufficiently large Z will bind positrons. Correspondingly, it is expected that as Z increases the Coulomb barrier will become increasingly transparent to positrons. This is an example of Klein tunnelling.

I. THE KLEIN PARADOX

Klein [1] considered the reflection and transmission of electrons of energy E incident on the potential step $V(x) = V, x > 0; V(x) = 0, x < 0$ for $E < V$ in the one-dimensional time-independent Dirac equation which can be written [2] in terms of the usual Pauli matrices as

$$(\sigma_x \frac{\partial}{\partial x} - \sigma_z (E - V(x)) + m)\psi(x) = 0 \tag{1}$$

He found that the reflection and transmission coefficients R_S, T_S when the step height $V > 2m$ were given by

$$R_S = \left(\frac{1 - \kappa}{1 + \kappa}\right)^2 \qquad T_S = \frac{4\kappa}{(1 + \kappa)^2} \tag{2}$$

where κ is the kinematic factor

$$\kappa = \frac{p}{k} \frac{E + m}{E + m - V} = \sqrt{\frac{(V - E + m)(E + m)}{(V - E - m)(E - m)}} \tag{3}$$

$k = \sqrt{E^2 - m^2}$ is the momentum of the incident electron and $p = -\sqrt{(V - E)^2 - m^2}$ is the momentum of the transmitted particle for $x > 0$. Provided p is negative[1], R_S and T_S are both positive and satisfy $R_S + T_S = 1$. Telegdi [3] has recently reviewed the history of the Klein paradox, concentrating on the inaccuracies which followed from choosing the wrong sign of p. Nowadays the paradox is taken to lie in the result that as $V \to \infty$ for fixed E, T_S tends to a non-zero limit. The physical essence of the Klein paradox thus lies in the prediction that fermions can pass through large repulsive potentials without exponential damping. This we call Klein tunnelling.

There have been various attempts over the last seventy years to explain the Klein paradox. The preferred explanation given by Telegdi and in textbooks [4] is due to Hansen and

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[1]Pauli pointed out to Klein [1] the necessity of choosing p to be negative so that the group velocity of the particle for $x > 0$ was positive.

Ravndal [5]: the left-right asymmetric step emits electron-positron pairs just as a constant electric field does [6] and it is the inability of an observer to distinguish between this emission and the static scattering properties of the step which gives rise to the paradox. T_S measures the positrons emitted by the step while R_S is not unity because the Pauli principle causes destructive interference between the emitted electron and the reflected electron. In the absence of particle production R_S would indeed be 1 and T_S would be 0.

These claims are investigated in this paper. We begin from a viewpoint which sees no connection between left-right asymmetry and quantum tunnelling, nor indeed any necessary connection between either of these and particle production. We will seek to show that Klein tunnelling is characteristic of relativistic wave equations as such, because negative energy states really do represent physical particles. Our approach in this respect is similar to that of Jensen et al [7]. We therefore investigate a class of problems in which Klein tunnelling takes place even in the absence of particle production. Klein was unfortunate in that he considered a pathological problem which gives misleading results: the potentials we consider can be very similar to a Klein step but are better defined and both their time-independent and emission properties can be found simply using Eq.(1) in the presence of a four-potential $V(x)$. Particle emission from strong potentials is well-understood in terms of supercritical positron production [4]. A full field theoretic treatment is given in our earlier paper [2] but provided we restrict ourselves to potentials which are switched on adiabatically we manage here with first quantisation (except in the Appendix where we derive the relation between the transmission coefficient T_S and the current emitted by the step). Finally we look at Klein tunnelling in the context of the Coulomb potential.

II. SCATTERING BY A SQUARE BARRIER

Consider the square barrier $V(x) = V, |x| < a; V(x) = 0, |x| > a$ instead of the Klein step. For $ma >> 1$ this potential may be expected to give similar results to those found by Klein.

It does, but the results are perhaps surprising.

It is easy to show that the reflection and transmission coefficients for the square barrier are given by [7]

$$R = \frac{(1 - \kappa^2)^2 \sin^2(2pa)}{4\kappa^2 + (1 - \kappa^2)^2 \sin^2(2pa)} \tag{4}$$

$$T = \frac{4\kappa^2}{4\kappa^2 + (1 - \kappa^2)^2 \sin^2(2pa)} \tag{5}$$

Note that Klein tunnelling is enhanced for a barrier: if

$$2pa = N\pi \tag{6}$$

corresponding to $E_N = V - \sqrt{m^2 + N^2\pi^2/4a^2}$ the electron passes right through the barrier with no reflection: this is called a transmission resonance [2]. As a becomes very large for fixed m, E and V, we can average over the phase angle pa to find the limit

$$R_\infty = \frac{(1 - \kappa^2)^2}{8\kappa^2 + (1 - \kappa^2)^2} \qquad T_\infty = \frac{8\kappa^2}{8\kappa^2 + (1 - \kappa^2)^2} \tag{7}$$

It may seem unphysical that R_∞ and T_∞ are not the same as R_S and T_S but it is not: it is well known in electromagnetic wave theory [12] that reflection off a transparent barrier of large but finite width (with 2 sides) is different from reflection off a transparent step (with 1 side). R_∞ and T_∞ thus involve Klein tunnelling but they arise from a more physical problem than the Klein step. The zero of potential is properly defined for a barrier whereas it is arbitrary for a step and the energy spectrum of a barrier is easily calculable. Emission from a barrier or well is described by supercriticality: the condition when the ground state energy of the system overlaps with the continuum ($E = m$ for a barrier; $E = -m$ for a well) and so any connection between particle emission and the time-independent scattering coefficients R and T can be investigated.

III. FERMIONIC EMISSION FROM A SQUARE WELL/BARRIER

We discussed this topic in our previous paper [2] which we refer to as CDI. We quickly review the argument. Spontaneous fermionic emission is a non-static process and in the case of a seemingly static potential, it is necessary to ask how the potential was switched on from zero. We follow CDI in turning on the potential adiabatically.

The bound state spectrum for the well $V(x) = -V, |x| < a; V(x) = 0, |x| > a$ is easily obtained [2]: there are even and odd solutions given by the equations

$$\tan pa = \sqrt{\frac{(m-E)(E+V+m)}{(m+E)(E+V-m)}} \tag{8}$$

$$\tan pa = -\sqrt{\frac{(m+E)(E+V+m)}{(m-E)(E+V-m)}} \tag{9}$$

where now the well momentum is given by $p^2 = (E+V)^2 - m^2$. We have changed the sign of V so that it is now attractive to electrons rather than positrons in order to conform with other authors who have studied supercritical positron emission rather than electron emission .

A. Narrow Well

The simplest case to discuss is a very narrow deep well $V(x) = -\lambda\delta(x)$ which is the limit of a square well with $\lambda = 2Va$. The bound states are then given for even (e) and odd (o) wave functions by

$$E = m\cos\lambda \quad (e) \qquad E = -m\cos\lambda \quad (o) \tag{10}$$

When the potential is initially turned on and λ is small there is one bound state just below $E = m$. As λ increases, E decreases and at $\lambda = \pi/2$, E reaches zero. For $E > \pi/2$, E

becomes negative and if the well were originally vacant, we now have the absence of a negative energy state which we must interpret as the presence of a (bound) positron according to Dirac's hole theory. We can use the anticommutation relations for the electron field [2] to write the charge Q as $Q = Q_0 + Q_p$ where Q_0 is a c-number called the vacuum charge [10] [11] and Q_p is the particle (or normal-ordered) charge. The total charge Q is always conserved. For λ just larger than $\pi/2$, $Q_p = +1$ because of the presence of the positron and so the vacuum charge Q_0 must now equal -1 to conserve charge. As the potential is increased further, λ will reach π. Here $E = -m$ which is the condition for supercriticality: the bound positron reaches the continuum and becomes free. This is the well-known scenario of spontaneous positron production first discussed [8] [9] over 25 years ago. Note that at supercriticality $\lambda = \pi$, the even bound state disappears and the first odd state appears.

We can continue to increase λ and count positrons: the total number of positrons produced for a given λ is the number of times E has crossed $E = 0$; that is

$$Q_p = Int\,[\frac{\lambda}{\pi} + \frac{1}{2}] \qquad (11)$$

where $Int[x]$ denotes the integer part of x. The more interesting quantity for us is the number of supercritical positrons Q_S: the number of states which have crossed $E = -m$. This is given by

$$Q_S = Int\,[\frac{\lambda}{\pi}] \qquad (12)$$

Note that for any λ there is at most one bound positron state.

B. Wide Well

We consider the general case of $V > 2m$ and then look in particular at the case $ma \gg 1$ most closely corresponding to the Klein step. We must find first the condition for supercriticality of the potential V, and then the number of bound and supercritical positrons produced for a

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given V. From Eq (8) we see that the ground state becomes supercritical when $pa = \pi/2$ and therefore $V_1^c = m + \sqrt{m^2 + \pi^2/4a^2}$. From Eq (9) the first odd state becomes supercritical when $pa = \pi$ and $V_2^c = m + \sqrt{m^2 + \pi^2/a^2}$. Clearly the supercritical potential corresponding to the Nth positron is

$$V_N^c = m + \sqrt{m^2 + N^2\pi^2/4a^2} \tag{13}$$

It follows from Eq (13) that $V = 2m$ is an accumulation point of supercritical states as $ma \to \infty$. Furthermore it is a threshold: a potential V is subcritical if $V < 2m$. It is not difficult to show for a given $V > 2m$ that the number of supercritical positrons is given by

$$Q_S = Int[(2a/\pi)\sqrt{V^2 - 2mV}] \tag{14}$$

The corresponding value of the total positron charge Q_p can be shown using Eqs (8,9) to satisfy

$$Q_p - 1 \le Int[(2a/\pi)\sqrt{V^2 - m^2}] \le Q_p \tag{15}$$

so for large a we have the estimates

$$Q_p \sim (2a/\pi)\sqrt{V^2 - m^2}; \quad Q_S \sim (2a/\pi)\sqrt{V^2 - 2mV} \tag{16}$$

Now we can build up an overall picture of the wide square well $ma >> 1$. When V is turned on from zero in the vacuum state an enormous number of bound states is produced. As V crosses m a very large number Q_p of these states cross $E = 0$ and become bound positrons. As V crosses $2m$ a large number Q_S of bound states become supercritical together. This therefore gives rise to a positively charged current flowing from the well. But in this case, unlike that of the Klein step, the charge in the well is finite and therefore the particle emission process has a finite lifetime. Nevertheless, for ma large enough the transient positron current for a wide barrier is approximately constant in time for a considerable time as we shall see in the next section.

IV. EMISSION DYNAMICS

We now restrict ourselves to the case $V = 2m + \Delta$ with $\Delta << m$. This is not necessary but it avoids having to calculate the dynamics of positron emission while the potential is still increasing beyond the critical value. We can assume all the positrons are produced almost instantaneously as the potential passes through $V = 2m$. It also means that the kinematics are non-relativistic. Hence for a sufficiently wide well so that Δa is large, $Q_S \sim 4\Delta a/\pi$. The well momentum of the Nth supercritical positron is still given by Eq (6) $p_N a = N\pi/2$ which corresponds to an emitted positron energy $|E_N| = 2m + \Delta - \sqrt{p_N^2 + m^2} > m$. Note that the emitted energies have discrete values although for a large, they are closely spaced.

The lifetime τ of the supercritical well is given by the time for the slowest positron to get out of the well. The slowest positron is the deepest lying state with $N = 1$ and momentum $p_1 = \pi/2a$. Hence $\tau \approx ma/p_1 = 2ma^2/\pi$. So the lifetime is finite but scales as a^2. But a large number of positrons will have escaped well before τ. There are Q_S supercritical positrons initially and their average momentum $\overline{p}$ corresponds to $N = Q_S/2$; hence $\overline{p} = \Delta$ which is independent of a. Thus a transient current of positrons is produced which is effectively constant in time for a long time of order $\overline{\tau} = ma/\Delta$. We thus see that the square well (or barrier) for a sufficiently large behaves just like the Klein step: it emits a seemingly constant current with a seemingly continuous energy spectrum. But initially the current must build up from zero and eventually must return to zero. So the well/barrier is a time-dependent physical entity with a finite but long lifetime for emission of supercritical positrons or electrons. In the Appendix we show that if we assume that the transient current emitted is constant in time (which it is not), then it is possible to obtain a relationship between this current and the transmission coefficient just as for the Klein step.

Note again that the transmission resonances of the time-independent scattering problem coincide with the energies of particles emitted by the well or barrier. It is therefore tempting

to use the Pauli principle to explain the connection. Following Hansen and Ravndal, we could say that R must be zero at the resonance energy because the electron state is already filled by the emitted electron with that energy. But it is easy to show that the reflection coefficient is zero for bosons as well as fermions of that energy, and no Pauli principle can work in that case. Furthermore emission ceases after time τ whereas $R = 0$ for times $t > \tau$. It follows that we must conclude that Klein tunnelling is a physical phenomenon in its own right, independent of any emission process. In our next section we consider further ways to investigate this tunnelling theoretically and experimentally.

V. KLEIN TUNNELLING AND THE COULOMB BARRIER

It is clear from Eq (4) that while the reflection coefficient R cannot be 0, neither is the transmission coefficient T exponentially small for energies $E < V$, even though the scattering is classically forbidden. The simplest way to understand this is to consider the negative energy states under the potential barrier as corresponding to physical particles which can carry energy in exactly the same way that positrons are described by negative energy states which can carry energy. It follows from Eq (2) that R_S and T_S correspond to reflection and transmission coefficients in transparent media with differing refractive indices: thus κ is nothing more than an effective fermionic refractive index corresponding to the differing velocities of propagation by particles in the presence and absence of the potential. On this basis, tuning the momentum p to obtain a transmission resonance for scattering off a square barrier is nothing more than finding the frequency for which a given slab of refractive material is tranparent. This is not a new idea. In Jensen's words "A potential hill of sufficient height acts as a Fabry-Perot etalon for electrons, being completely transparent for some wavelengths, partly or completely reflecting for others" [13].

We can now look in more detail at Klein tunnelling: both in terms of our model square well/barrier problem and at the analogous Coulomb problem. The interesting region is

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where the potential is strong but subcritical so that emission dynamics play no role and sensible time independent scattering parameters can be defined. For electrons scattering off the square barrier $V(x) = V$ we would thus require $V < V_1^c = m + \sqrt{m^2 + \pi^2/4a^2}$ together with $V > 2m$ so that positrons can propagate under the barrier. For the corresponding square well $V(x) = -V$ there are negative energy bound states $0 > E > -m$ provided that $V > \sqrt{m^2 + \pi^2/4a^2}$ [cf. Eq.(15)]. So when the potential well is deep enough, it will in fact bind positrons. Correspondingly, a high barrier will bind electrons. It is thus not surprising that electrons can tunnel through the barrier for strong subcritical potentials since they are attracted by those potential barriers. Another way of seeing this phenomenon is by using the concept of effective potential $V_{eff}(x)$ which is the potential which can be used in a Schrodinger equation to simulate the properties of a relativistic wave equation. For a potential $V(x)$ introduced as the time-component of a four-vector into a relativistic wave equation (Klein-Gordon or Dirac), it is easy to see that $2mV_{eff}(x) = 2EV(x) - V^2(x)$. Hence as the energy E changes sign, the effective potential can change from repulsive to attractive.

For the Coulomb barrier, Anchishkin [14] has already suggested looking at scattering of π^+ off heavy nuclei to see if there was any experimental evidence for tunnelling: the Klein-Gordon equation like the Dirac equation has negative energy solutions, so similar arguments apply. His calculations show a 20% enhancement in $\pi^+ - ^{238}U$ scattering compared with non-relativistic expectations. The analogous process for fermions would be positron-heavy nucleus scattering. For a positron of initial energy E incident on a heavy nucleus of charge Z the classical turning point $r_c = Z\alpha/E$. So it would be interesting to measure the wave function at the origin for positron scattering off a heavy nucleus compared with the wave function at the origin for electron scattering off the nucleus at the same energy in order to demonstrate tunnelling.

Provided that $Z\alpha < 1, E > 0$ and normal Coulomb wave functions should be accurate

enough for the calculation of the ratio $\rho = |\psi(0)|^2_{pos} \, / \, |\psi(0)|^2_{el}$ of wave functions at the origin. The wave function at the origin for a Dirac particle is singular but the ratio is finite. The exact continuum wave functions for a Dirac particle in a Coulomb potential are discussed by Rose [15]. We shall just write down his results: if the particles are non-relativistic then

$$\rho = e^{-2\pi Z\alpha E/p} \tag{17}$$

where p and E are the particle momenta and energies and this is ofcourse exponentially small as $p \to 0$. But if the particles are relativistic

$$\rho = f e^{-2\pi Z\alpha} \tag{18}$$

where f is a ratio of complex gamma-functions and is approximately unity for large Z. So $\rho \sim e^{-2\pi Z\alpha} \approx 10^{-3}$ for $Z\alpha \sim 1$ which is not specially small although it still decreases exponentially with Z. It should be possible to measure ρ by, for example, internal electron-positron pair production in positron-nucleus scattering versus pair production in electron-nucleus scattering. Alternatively if the positrons were longitudinally polarised any resultant asymmetry in the scattering cross section would be due to weak processes. These could only occur at a distance $r_w = 1/M_Z$ which is effectively at the origin for atomic systems. So the asymmetry for polarised positron scattering could be compared with that for polarised electron scattering. The ratios of the asymmetries would give ρ directly.

But while experiments for $Z\alpha < 1$ should show tunnelling it will not yet amount to Klein tunnelling. For that we require Z large enough so that bound positron states are present. This means that Z must be below its supercritical value Z_c of around 170 but large enough for the $1s$ state to have $E < 0$. The calculations of references [8] and [9] which depend on particular models of the nuclear charge distribution give this region as $150 < Z < Z_c$ which unfortunately will be difficult to demonstrate experimentally. Nevertheless, the theory seems to be clear: in this subcritical region positrons should no longer obey a tunnelling relation which decreases exponentially with Z such as that of Eq. (18). Instead the Coulomb

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barrier should become more transparent as Z increases, at least for some energies. The work of Jensen and his colleagues [7] shows that a transmission resonance (i.e. maximal transmission) for positron scattering on a Coulomb potential may well occur at $Z = Z_c$ although the onset of supercriticality implies that time independent scattering quantities will then no longer be well-defined. Further detailed calculations are needed to clarify the situation for positron scattering off nuclei with Z just below Z_c. It is also important to see if the square well/barrier relationship we illustrate above that transmission resonances at barriers occur when the corresponding well becomes supercritical [cf. Eqs (6) and (8) can be generalised to arbitrary potentials.

VI. CONCLUSIONS

We hope that this discussion has demonstrated that the Klein step is pathological and therefore a misleading guide to the underlying physics. It represents a limit in which time-dependent emission processes become time-independent and therefore a relationship between the emitted current and the transmission coefficient exists, as we show in the Appendix. In general no such relationship exists. The underlying physics discovered by Klein in his solution of the Dirac equation is not particle emission but tunnelling by means of the negative energy states which are characteristic of relativistic wave equations, similar to interband tunnelling in semiconductors [16]. It is time to finally bring this 70 year old puzzle to a conclusion.

VII. APPENDIX: THE CALCULATION OF THE VACUUM CURRENT IN THE PRESENCE OF A KLEIN STEP.

Consider the potential step $V(x) = V, x > 0; V(x) = 0, x < 0$ for $V > 2m$ of section I. We will show that the expectation value of the current in the vacuum state in the presence of the step is non-zero. The derivation hinges on a careful definition of the vacuum state.

A. The normal modes in the presence of the Klein step.

A properly-normalised positive energy solution to the Dirac equation (1) can be written

$$\sqrt{\frac{E+m}{2k}} \begin{pmatrix} i \\ \dfrac{k}{E+m} \end{pmatrix} e^{ikx} \tag{19}$$

Scattering is usually described by a solution describing a wave incident (say from the left) plus a reflected wave (from the right) plus a transmitted wave (to the right). It is convenient here to use waves of different form either describing a wave (subscript L) incident from the left with no reflected wave or describing a wave (subscript R) incident from the right with no reflected wave. Positive and negative energy wavefunctions will be denoted by u and v respectively. It is clear that the nontrivial result we are seeking arises from the overlap of the negative energy continuum $E < V - m$ on the right with the positive energy continuum $E > m$ on the left. We are thus concerned with wavefunctions with energies in the range $m < E < V - m$. The expressions for u_L, u_R in this energy range are given below.

$$u_L(E, x) = \frac{\sqrt{2\kappa}}{\kappa + 1} \sqrt{\frac{E+m}{k}} \begin{pmatrix} i \\ \dfrac{k}{E+m} \end{pmatrix} e^{ikx} \theta(-x) +$$

$$\left\{ \frac{\kappa - 1}{\kappa + 1} \sqrt{\frac{V - E - m}{2|p|}} \begin{pmatrix} i \\ \dfrac{|p|}{E+m-V} \end{pmatrix} e^{i|p|x} + \sqrt{\frac{V - E - m}{2|p|}} \begin{pmatrix} i \\ \dfrac{-|p|}{E+m-V} \end{pmatrix} e^{-i|p|x} \right\} \theta(x) \tag{20}$$

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$$u_R(E,x) = \left\{ \frac{1-\kappa}{1+\kappa}\sqrt{\frac{E+m}{2k}} \begin{pmatrix} i \\ k \\ \overline{E+m} \end{pmatrix} e^{ikx} + \sqrt{\frac{E+m}{2k}} \begin{pmatrix} i \\ -k \\ \overline{E+m} \end{pmatrix} e^{-ikx} \right\} \theta(-x) +$$

(21)

$$+ \frac{\sqrt{2\kappa}}{\kappa+1} \begin{pmatrix} i \\ |p| \\ \overline{E+m-V} \end{pmatrix} e^{i|p|x}\theta(x)$$

We write $|p|$ rather than p in these equations since as Pauli noted, the group velocity is negative for $x > 0$. For $V - E = \sqrt{p^2 + m^2}$ we have $dE/dp = - |p| / \sqrt{p^2 + m^2}$. This gives the expressions above.

We need to evaluate the currents corresponding to the solutions of Eqs (20,21). According to our conventions $\alpha_x \equiv \gamma_0 \gamma_x = -\sigma_y$ so

$$j_L \equiv -u_L^\dagger(E,x)\sigma_y u_L(E,x) = \frac{2\kappa}{\pi(\kappa+1)^2} \tag{22}$$

$$j_R \equiv -u_R^\dagger(E,x)\sigma_y u_R(E,x) = -\frac{2\kappa}{\pi(\kappa+1)^2} \tag{23}$$

B. The definition of the vacuum and the vacuum expectation value of the current.

We expand ψ in terms of creation and annihilation operators terms in terms of our left- and right-travelling solutions:

$$\psi(x,t) = \int dE \{ a_L(E)u_L(E,x)e^{-iEt} + a_R(E)u_R(E,x)e^{-iEt} +$$

(24)

$$+ b_L^\dagger(E)v_L(E,x)e^{iEt} + b_R^\dagger(E)v_R(E,x)e^{iEt} \}$$

with $\psi^\dagger$ given by the Hermitean conjugate expansion. We must now determine the appropriate vacuum state in the presence of the step. States described by wavefunctions $u_L(E,x)$ and $v_L(E,x)$ correspond to (positive energy) electrons and positrons respectively coming from the left. Hence with respect to an observer to the left (of the step) such states should be absent from the vacuum state, so

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$$a_L(E) \ket{0} = 0, \; b_L(E) \ket{0} = 0 \tag{25}$$

Wavefunctions $u_R(E, x)$ for $E > m + V$ describe for an observer to the right, electrons incident from the right. These are not present in the vacuum state hence

$$a_R(E) \ket{0} = 0 \text{ for } E > m + V \tag{26}$$

Wavefunctions $v_R(E, x)$ describe, again with respect to an observer to the right, positrons incident from the right; again

$$b_R(E) \ket{0} = 0 \tag{27}$$

The wavefunctions that play the crucial role in the Klein problem belong to the set $u_R(E, x)$ for $m < E < V - m$. For an observer to the right these states are positive energy positrons and hence they should be filled in the vacuum state, i.e.

$$a_R^\dagger(E) a_R(E') \ket{0} = \delta(E - E') \ket{0} \; , \; m < E < V - m \tag{28}$$

Having specified the vacuum the next and final step is the calculation of the vacuum expectation value of the current:

$$\bra{0} j \ket{0} = \frac{1}{2} \left(- \bra{0} \psi^\dagger \sigma_y \psi \ket{0} + \bra{0} \psi \sigma_y \psi^\dagger \ket{0} \right) \tag{29}$$

Substituting (24) in (29) and noticing that all terms involving v_L and v_R can be dropped since the corresponding energies lie outside the interesting range $m < E < V - m$ we end up with

$$\bra{0} j \ket{0} = -\tfrac{1}{2} \int dE dE' \{ \bra{0} a_L^\dagger(E) a_L(E') \ket{0} u_L^\dagger(E, x) \sigma_y u_L(E', x) +$$

$$+ \bra{0} a_L(E) a_L^\dagger(E') \ket{0} u_L^\dagger(E', x) \sigma_y u_L(E, x) - \bra{0} a_R^\dagger(E) a_R(E') \ket{0} u_R^\dagger(E, x) \sigma_y u_R(E', x) +$$

$$+ \bra{0} a_R(E) a_R^\dagger(E') \ket{0} u_R^\dagger(E', x) \sigma_y u_R(E, x) \}$$

$$\tag{30}$$

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The first term in (30) vanishes due to (25). The second term becomes

$u_L^\dagger(E',x)\sigma_y u_L(E,x)\delta(E-E')$ if we use the anticommutation relations and (25). The third

term yields $-u_R^\dagger(E,x)\sigma_y u_R(E,x)\delta(E-E')$ using (28) and the fourth term vanishes using

the anticommutation relations (i.e. the exclusion principle; the state $|0\rangle$ already contains

an electron in the state u_R hence we get zero when we operate on it with $a_R^\dagger$). One energy

integration is performed immediately using the δ function. The final result is

$$\langle 0|\, j\, |0\rangle = \frac{1}{2}\int dE(-j_L + j_R) = -\int dE\frac{4\kappa(E)}{(\kappa(E)+1)^2} \equiv -\int dET_S(E) \tag{31}$$

where the energy integration is over the Klein range. This is the result first obtained by

Hansen and Ravndal [5] linking the pair production current with the transmission coefficient.

But the fact that these quantities are linked does not represent a causal relationship between

them. This is clear if we attempt to repeat this analysis for a square barrier for $V > 2m$

and $ma >> 1$. It is easily shown by the same method that

$$\langle 0|\, j\, |0\rangle = -\int dE\frac{8\kappa^2}{(1-\kappa^2)^2 + 8\kappa^2} \equiv -\int dET_\infty(E) \tag{32}$$

provided that averages over phase angles are understood, where now the current is measured

by an observer situated at a point $x > a$. The result of Eq. (32) of course can only be true

in the approximation where the transient current is taken to be independent of time.

Note also that whereas for a Klein step we describe the particle emission in terms of electron-

positron pair production while we say that a supercritical well will spontaneously emit

positrons. These terms are relative: an observer outside a supercritical well would see a

positron current going to the right while an observer measuring the same phenomenon but

inside the well would talk of an electron current going to the left.

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